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On turbulent entrainment at a stable density interface

By L. H. KANTHA, O. M. PHILLIPS AND R. S. AZAD†

Department of Earth and Planetary Sciences, The Johns Hopkins University, Baltimore, Maryland 21218

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Turbulent entrainment at the density interface of a stable two-layer stratified fluid is studied in the laboratory, a constant surface stress being applied at the free surface. Conservation of mass requires that the overall Richardson number $Ri = Dg\delta\rho/\rho u_*^2$ is constant in each experiment, where D is the depth of the mixed layer, $g\delta\rho/\rho$ the buoyancy difference and u_* the friction velocity. If the entrainment rate $E = u_e/u_*$ is a function only of Ri, it is therefore constant in each experiment and can be measured with a greater accuracy than has previously been attained. The functional dependence of u_e/u_* on Ri is established over the range $30 < Ri < 1000$; it is found not to follow any simple power law. The entrainment rates are considerably higher than those measured by Kato & Phillips (1969), for which the fluid below the mixed layer was linearly stratified. Such a condition allows internal gravity waves to be radiated downwards and the reduction in entrainment rate is consistent with that found by Linden (1975).

1. Introduction

Turbulent mixing by entrainment across a stable density interface is a common process in a variety of geophysical flows. Turbulence in the upper layer of the ocean or of a lake, generated directly by the wind or by wind-induced shear, penetrates into denser, more quiescent fluid below. Similarly, in the atmosphere a low level, highly stable inversion frequently forms at the top of a mixed layer above the ground. In each case, the entrainment must proceed against a stable density jump.

A number of pertinent laboratory experiments of two types have been undertaken to study this process. In the first kind, pioneered by Rouse & Dodu (1955) and later investigated in detail by Turner's group in Cambridge, the turbulence is produced by one or two oscillating grids in a tank filled with stratified fluid. The result is the appearance of a stable density interface separating an essentially homogeneous turbulent layer of fluid from almost quiescent fluid (with a single stirrer) or from another quasi-homogeneous turbulent layer (with two stirrers). In these experiments there is no significant mean flow or shear. The presence of a shear across the interface is very common in the atmosphere and

† Present address: Department of Mechanical Engineering, University of Manitoba, Winnipeg, Manitoba, Canada.

the ocean, so that it is difficult to apply results from these experiments directly to many field situations. Nevertheless, valuable information has been obtained by Crapper & Linden (1974), Cromwell (1960), Thompson & Turner (1975), Turner (1968, 1973) and Wolanski & Brush (1975), though there remain a number of discrepancies. Turner found that $E = u_e/u_* \propto Ri^{-1}$ at low Richardson numbers but for higher values of Ri, $E \propto Ri^{-1}$ when the stratification was produced by temperature differences and $E \propto Ri^{-\frac{3}{2}}$ when it was the result of salinity differences. Although this apparently implies that the entrainment rate depends on the Péclet number $Pe = lu/k$, where k is the molecular diffusivity, later measurements of the interfacial structure by Crapper & Linden (1974) seem to indicate that at high enough Péclet numbers the entrainment should not be influenced by molecular diffusivity. Experiments of this kind have been described in the recent comprehensive monograph by Turner (1973). (See also Kantha 1975.)

The second type of experiment involves shear-flow turbulence produced either by a surface stress in an annular tank (as in Kato & Phillips 1969) or by the momentum flux from a series of jets directed along the wall as in the race-track apparatus of Moore & Long (1971). Each of these studies suggested that $E \propto Ri^{-1}$, though the scatter, particularly in the Kato–Phillips experiment, was considerable. In these experiments, the turbulence in the mixed layer is partially or totally supported by shear and it is not evident that results from stirring experiments are germaine to this situation. Not only is the turbulent structure in the shear layer different, but the mechanisms involved in the entrainment itself are different in the two types of experiment.

In the stirring experiments, visual observations indicate that the eddies produced by the grid impinge upon the interface, setting up a pattern of irregular interfacial standing waves, which sporadically break, resulting in the ejection of thin filaments of fluid from the denser into the lighter layer, with subsequent mixing. The entrainment may also proceed by the mechanism of deformation of the interface by eddies, subsequent recoil and ejection of the fluid from the denser region across the interface as proposed by Linden (1973). In contrast, the experiments with shear at a low overall Richardson number produce a convoluted interface not unlike that in a constant-density flow such as that at the outside edge of a boundary layer. At higher values of the Richardson number, the amplitude of the convolutions decreases but if the upper fluid is dyed, the interfacial region is observed to contain sporadic groups of rolling eddies at the base of the turbulent region, not unlike those observed by Thorpe in the instability of a stratified shear layer. The flow is, of course, highly turbulent; nevertheless, these eddies occur intermittently in quite regular groups of four or six. At even higher Richardson numbers, the interface is almost flat: the convolutions and eddies have gone and it looks more like an air–water interface ruffled (in slow motion) as if by the wind, but with occasional wisps of the lower fluid being drawn up.

The experiment described here involves a very simple but significant improvement on the Kato–Phillips experiment. The same apparatus is used but instead of an initially uniformly stratified fluid, the tank initially contains two homogeneous layers with a density difference $\delta\rho$. As the entrainment proceeds, conservation of mass requires that $\delta\rho D$ is constant, so that in a given experiment, and with u_* held fixed, the overall Richardson number is constant. In the absence of extraneous effects, the entrainment velocity u_e is also constant and the depth of the mixed layer a linear function of time whose slope can be measured with reasonable accuracy. The experiment is then self-checking, departures from linearity in $D(t)$ as a function of time indicating the onset of unwanted side effects, as described in detail below. Furthermore, the fact that the lower layer is homogeneous prevents the radiation of internal gravity waves from the moving interface, a source of energy loss that is difficult to measure with accuracy. It can, however, be substantial, as Linden (1975) showed in a stirring experiment of the Turner type. A disadvantage of the present experiment is that each run provides a single point on the curve $u_e/u_* = f(Ri)$, so that to define this curve, the experiment must be repeated a number of times.

2. The experiment

The present investigation uses the same apparatus as that of Kato & Phillips (1969). Its salient features will be described briefly, but for a more detailed discussion of the experimental arrangement and the procedures used, the reader is referred to the paper by Kato & Phillips.

The annular tank used in the investigations, illustrated in figure 1, has inner and outer diameters of 106·7 cm and 152·4 cm respectively and is mounted on a horizontal base. The channel cross-section is 22·8 cm wide and 28 cm deep. Observations of the interface were made through a Plexiglas window in the outer wall of the tank. The interface was illuminated by a narrow slit of light (0·5 × 4 cm) through a glass window in the bottom of the tank. This beam illuminated only the centre of the channel cross-section and the interface was photographed by a 16 mm movie camera through the Plexiglas window in the side wall.

The surface stress was applied by means of a rotating screen at the surface, the total torque being measured by the extension of a light spring. Allowance was made for the friction in the wheel bearings; at the angular velocities normally used in the experiment, this was found to be constant and equal to 6 % in the worst case. The shear stress applied to the water surface is then

$$\tau = \rho u_*^2 = (T_\theta - T_F)/AR,$$

where T_θ is the torque indicated by the deflexion of the turntable, T_F is the frictional torque in the bearings, A is the surface area of the tank and R the mean radius. The friction velocity u_* is then $(\tau/\rho)^{\frac{1}{2}}$, where ρ is the density of fresh water.

A two-layer stratified system was produced by first filling the tank to the required depth with fresh water mixed with Flourescin dye, and then letting salt solution flow slowly into the tank through a hole in the bottom. The hole was capped by a thin flat plate so that the salt solution was forced to emerge only horizontally, under the fresh-water layer. The salt solution gradually lifted the

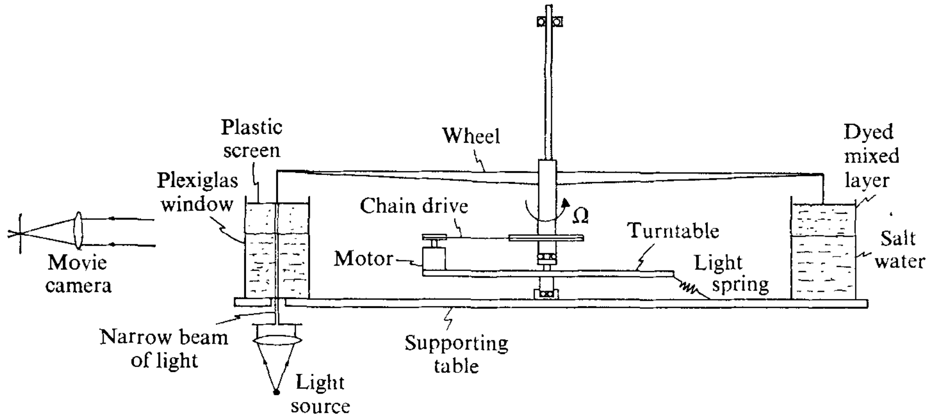

FIGURE 1. A cross-sectional sketch of the experimental apparatus. A photograph is given in Kato & Phillips (1969).

fresh-water layer towards the screen as filling continued. It usually took slightly more than an hour (the corresponding diffusion length scale being approximately 0·2 cm) to fill the tank and it was found that a reasonably sharp interface could be obtained by this method. The salt-water inflow was stopped when the screen was just immersed in the fresh-water layer at the top, with the cross-struts clear of the water. Because of the somewhat poorer condition of the equipment, it was not possible to adjust the height of the screen and maintain it horizontal to within any better than $\pm$ 0·5 cm, as opposed to the $\pm$ 0·2 cm indicated by Kato & Phillips. Also, the clearances between the edge of the screen and the sides of the tank were somewhat larger than the 0·4 cm quoted by them, and less uniform.

The interface between the light fresh-water layer at the top and the heavier salt solution at the bottom was visible as the result of the Fluorescin dye in the fresh water. The dye emits green radiation when illuminated by the slit of light. The light beam passes through the bottom layer with little scattering, and on encountering the dye in the mixed layer, the scattered light intensity undergoes a distinct jump at the interface. This is recorded on film by the movie camera, and the bottom of the interface is clearly and distinctly visible in the photographs. The motion of the wheel carrying equidistant light-reflecting marks was also visible in the movie photographs, so that the average speed of the screen could be found as a function of time. A centimetre scale mounted on the inner wall of the Plexiglas window and a stop-watch in the field of view of the camera aided the determination of the depth of the mixed layer and the angular velocity of the screen as a function of time. The procedure was to start the motor and the stop-watch with the camera on and to increase the motor speed smoothly but quickly until the desired angular deflexion of the turntable was attained. The motor was then constantly and continuously adjusted by manipulation of the speed control knob of the motor controller in order to maintain a constant turntable deflexion. Since the frictional loss in the wheel bearings is practically constant at the angular velocities of interest and the correction for inertia of the screen is negligible, the result is the application of a constant stress at the top of the fluid throughout the experiment. The magnitude of the applied stress during the few seconds it took to start the system and bring it to the desired running speed is, however, uncertain but the starting time was never more than 2 % of the average overall duration of the experiment, even at the lowest Richardson numbers.

The movies were projected on a screen with a 16 mm movie editor and read to determine the depth D of the mixed layer and the angular velocity Ω of the screen as functions of time. Because of the large convolutions of the interface at low Richardson numbers, the depth D was measured as an average over a few frames for each value, but in spite of this, the error in the determination under these conditions is higher and the scatter larger than at high Richardson numbers. Above a Richardson number of about 200, it was possible to read off the position of the interface visually during the experiment. Motion pictures were also taken of these cases in order to determine the screen velocities.

Experiments were made at values of Ri from about 30 to 1100, where $Ri = D_i g \delta\rho_i / \rho u_*^2$, where D_i and $\delta\rho_i$ are the initial depth of the mixed layer and the initial density jump across the interface. From conservation of mass, $D\Delta\rho$ remains constant throughout the experiment and because a constant shear stress is applied and maintained during the experiment, Ri remains constant throughout the experiment. Measurements at different Richardson numbers required repitition of the experiment and it was important to have as large a range of Ri as possible. The apparatus was difficult to control well if the friction velocity was less than 1·4 cm/s and it was felt inadvisable to use an initial depth of more than 5·4 cm because of the possible influence of aspect ratio, the ratio of width to depth. As a result, higher values of Ri could be attained only by an increase in the initial density contrast $\delta\rho_i/\rho$. With the use of common salt, the maximum density difference that could be attained was limited to about 0·2 and the maximum Richardson number to about 500. The only recourse was to use a salt of higher solubility and $CaCl_2$ was found to give values of $\delta\rho/\rho$ of up to 0·4. There are, however, several difficulties in the use of such a large density contrast. Variations in fluid inertia are not insignificant and the Boussinesq approximation becomes inaccurate. Moreover, the molecular viscosity of the solution increases (as shown in figure 2) with higher salt concentrations but only two experiments were conducted with the solutions initially almost saturated. In all cases, the minimum Reynolds number was high enough for the mixed layer to be fully turbulent. Some simple considerations below suggest that the variations in fluid inertia may produce an error of approximately 20 % at most; the agreement between experiments at the same Richardson number but different $\delta\rho/\rho$ suggests that this upper limit is probably generous.

Experiments were performed at friction velocities of 1·41 and 1·93 cm/s (corrected for friction in the wheel bearings). In most of the experiments, the initial depth of the upper layer was 5·4 cm, but in some, D_i was 2·7 cm. Both sodium chloride and calcium chloride were used to produce density stratification, but the majority of the experiments were done with common salt. Although $CaCl_2$ was used mainly to produce higher Richardson numbers, some runs were made at values between 30 and 500 attainable with NaCl also, in order to compare

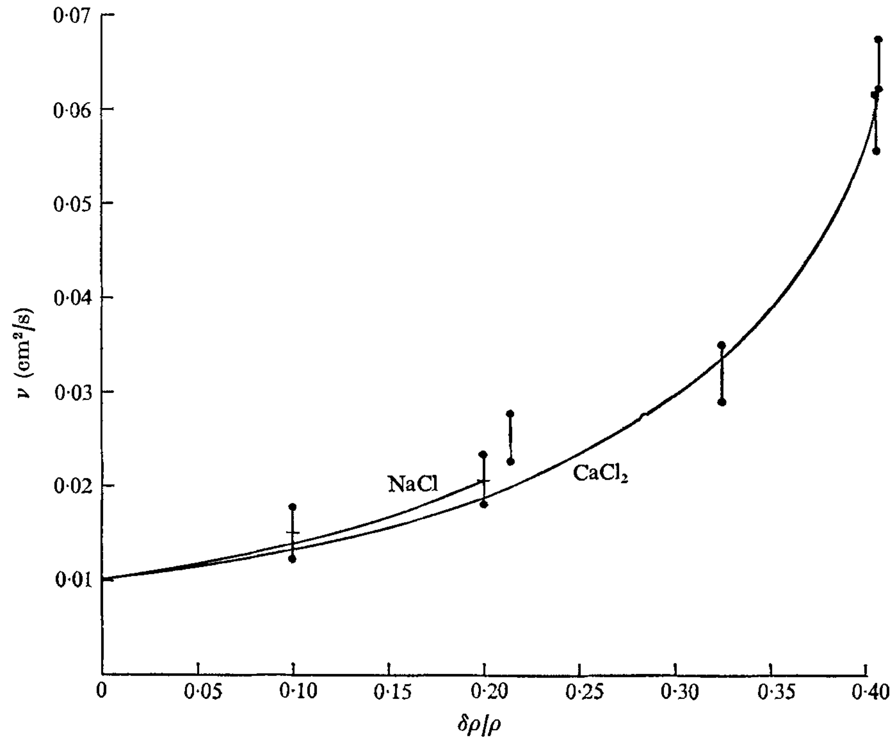

FIGURE 2. The variation of kinematic viscosity with $\delta\rho/\rho$ for sodium chloride and calcium chloride solutions at 20 °C, measured with a rotating-cylinder viscometer.

the entrainment rates for the two salts at the same Ri. Below $Ri = 300$, there was no detectable difference between the results; at $Ri = 500$ the calcium chloride gave entrainment rates slightly, though probably significantly, smaller than NaCl. With the exception of runs at $\delta\rho_i/\rho = 0{\cdot}31$ and 0·4, the kinematic viscosity of the heavier salt layer was never more than twice that of pure water. At $\delta\rho_i/\rho = 0{\cdot}31$ it was approximately three times that of water and in two experiments ($Ri = 1090$) at $\delta\rho_i/\rho = 0{\cdot}4$ the kinematic viscosity was abnormally high, about six times that of water.

The period of revolution of the screen never fell below 9 s, except for a few seconds at the start, and the correction for friction in the bearings was approximately constant, equivalent to a shear stress of $0{\cdot}26 \pm 0{\cdot}03$ cgs units. In each experiment u_* was kept as nearly as possible constant and we estimate that even considering the errors in calibration and the small oscillations about the mean position of the turntable, after the initial acceleration, the u_* values cannot have an error of more than about 3–4 %. The acceleration period comprised only a small fraction of the total duration of each experiment, even at low Ri, and during this interval, the effective applied stress was somewhat smaller (by perhaps 10 %). This could be partly responsible for the slightly smaller entrainment rates observed initially in most of the experiments. However, the entrainment rates at a fixed aspect ratio of 0·25 were obtained both from experiments in which the initial depth was 2·7 cm ($D_i/W = 0{\cdot}12$) and from those in which D_i was 5·4 cm ($D_i/W = 0{\cdot}24$); in the latter case the entrainment rate was obtained by extrapolating back almost to the starting point. The good consistency between the values estimated in these different experiments suggested that the influence of the initial acceleration phase was small.

3. Results from the experiment

There were obvious differences in the nature of the flow among experiments in which the overall Richardson number $Ri = Dg\delta\rho/\rho u_*^2$ ranged from 30 to 1100. At lower values, the interface developed massive convolutions similar to those observed in a homogeneous fluid at the outer edge of a boundary layer or a jet and one might speculate that the entrainment processes by which non-turbulent fluid is incorporated into the turbulence are similar. At intermediate Richardson numbers, 100–300, the convolutions were still present but accompanying them were intermittent groups of more regular eddies in the interfacial region, transverse to the flow and apparently the result of a local instability of the Kelvin–Helmholtz type. The scale of these eddies was considerably smaller than the depth of the turbulent layer as a whole, indicating that the instability was indeed local and associated with the relatively thin region of higher static stability and high shear at the base of the turbulent layer. Groups of eddies could be seen to develop, roll up and disappear into the turbulent background, and after a time, another group would appear. The sporadic nature of their occurrence suggests that they arise in response to a transient decrease in the local Richardson number

$$J = \frac{g\delta\rho/\partial z}{\rho(\partial u/\partial z)^2}$$

near the interface sufficient to promote the instability, the growth of the eddies and subsequent redistribution of mass and momentum restoring local stability (in this sense). It was not possible in these experiments to estimate the velocity and density gradients in the interfacial region but Dr L. Wyatt has measured from films of the experiment the frequency of occurrence of these events and estimated their contribution to the total rate of entrainment. Her work will be published elsewhere.

With a further increase in Ri, the extent of the convolutions becomes smaller and eventually a state is reached in which the interface is so stable that the turbulent eddies above rarely possess sufficient kinetic energy to lift or draw fluid from the denser layer across the interface. The interface remains sharp, in essence flat but with transient ripples and occasional wisps of fluid being drawn up. If the interfacial thickness is ϵ, turbulent entrainment requires that

$$\rho u_*^2 \gtrsim \delta\rho g\epsilon,$$

i.e. that the kinetic energy resident in the turbulent eddies matches or exceeds the potential energy needed to lift a parcel of heavier fluid across the interface, or

$$Dg\delta\rho/\rho u_*^2 \lesssim D/\epsilon.$$

A lower limit to ϵ occurs when the transfer across the interface is purely diffusive, in which case $\epsilon \sim \kappa/u_e$, so that for turbulent entrainment

$$\frac{Dg\delta\rho}{\rho u_*^2} \lesssim \left(\frac{Du_*}{\kappa}\right)\left(\frac{u_e}{u_*}\right). \tag{1}$$

At the higher Richardson numbers of the experiments, this condition was not satisfied strongly and, as will be seen later, the entrainment rate drops off sharply, so that the experiment, in some cases, took up to ten hours to complete. Towards the end of such an experiment, an appreciable drift velocity was evident in the fluid below the interface, the diffusion of momentum preceding that of salt.

The results of the experiment can be classified with the use of dimensional analysis. The entrainment velocity $u_e = dD/dt$ must be a function of u_*, the depth D of the mixed layer, the density ρ of fresh water, the density difference $\delta\rho$ across the interface, the gravitational acceleration g, the molecular viscosity ν and the molecular diffusivity κ, together with the geometrical parameters of the apparatus itself, the width W, the mean radius of the annulus R and the total depth of fluid H. Consequently

$$\frac{u_e}{u_*} = f_1\left(\frac{g\delta\rho D}{\rho u_*^2}, \frac{\delta\rho}{\rho}, \frac{u_* D}{\nu}, \frac{\nu}{\kappa}, \frac{D}{W}, \frac{W}{R}, \frac{W}{H}\right). \tag{2}$$

The angular velocity Ω of the screen needed to maintain the stress, non-dimensionalized as $R\Omega/u_*$, can be expressed as a different function of the same variables. The dependence on the first group, the overall Richardson number, is the one sought in this investigation; as pointed out previously, in any one experiment this ratio is constant. As far as the geometrical factors are concerned, the ratio W/R was fixed by the apparatus, having been made as small as possible. For accurate modelling, the radius R should be infinite; the influence of this factor is unknown and can be established only by repeating the experiment with different equipment. The influence of the total depth H was probably insignificant since all measurements that were considered in the analysis were taken when D/H was small and the experiments terminated when motion was observed near the bottom. The ratio D/W varied throughout each experiment, so that the influence of this factor contributes to the variation in the ratio u_e/u_* throughout an experiment. For application of these results to field situations, we are interested in the limit $D/W \rightarrow 0$. Variations in this ratio are perhaps the most troublesome of the 'spurious' effects and reflect the varying fraction of the surface stress balanced by drag of the side walls, as well as the varying structure of the annular flow as the aspect ratio increases. To reduce this effect, measurements were classified at fixed values of D/W and the extrapolation back to zero is discussed later. The Reynolds number of the flow, expressed as $u_* D/\nu$, was sufficiently large that the upper layer was always fully turbulent, though not really large enough for us to be confident that molecular viscosity is quite unimportant. There was, however, no consistent dependence found of u_e/u_* on $u_* D/\nu$ or on ν/κ, both of which ratios were varied, so that their influence will be neglected.

The variations in fluid inertia between the upper and lower layers, which are

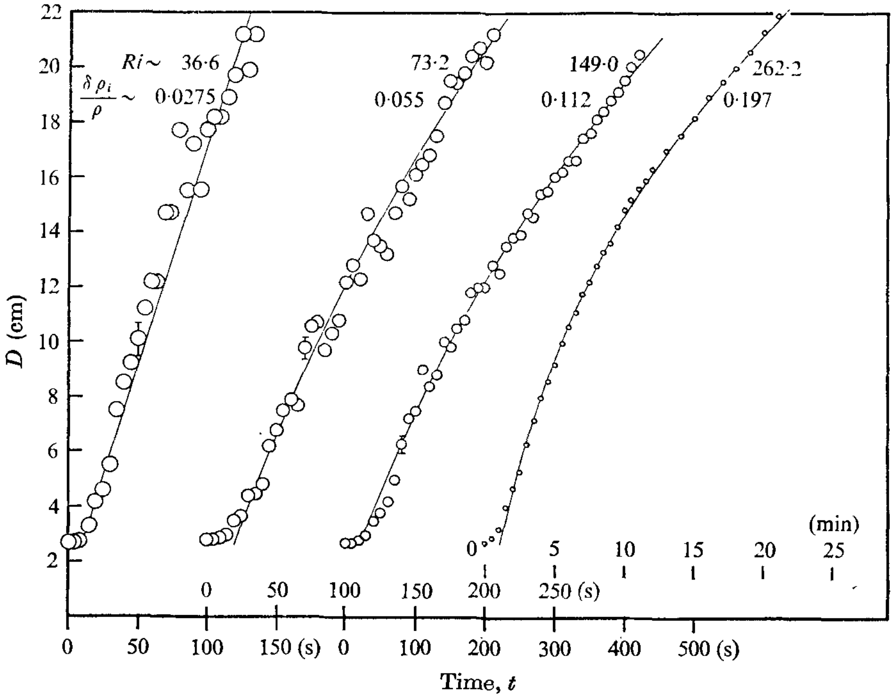

Figure 3. Depth D of the mixed layer as a function if time at various Richardson numbers with sodium chloride solution. $u_* = 1{\cdot}41$ cm/s, $D_i = 2{\cdot}7$ cm.

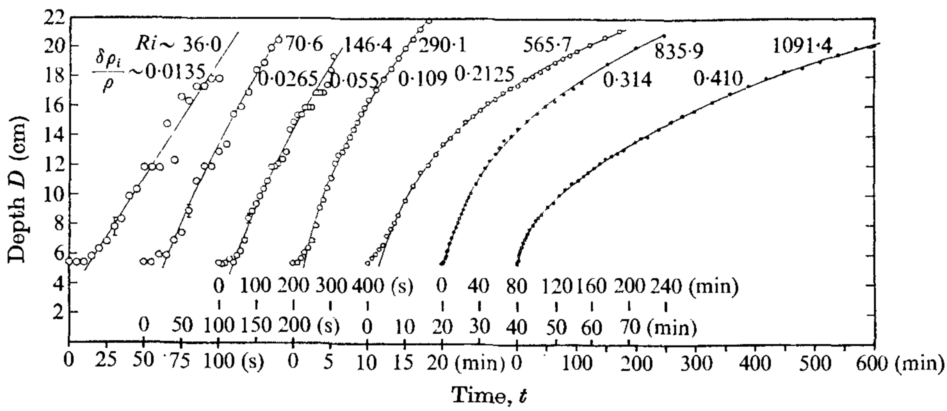

Figure 4. Depth D of the mixed layer as a function of time for measurements with calcium chloride solution. $u_* = 1{\cdot}41$ cm/s, $D_i = 5{\cdot}4$ cm.

neglected in the Boussinesq approximation, influence the experiment in two ways. First, at constant stress, the friction velocity u_* may be uncertain to within a factor $[1+(\Delta\rho/\rho)]^{\frac{1}{2}}$, where $\Delta\rho$ is the increase in density in the upper mixed layer above that of fresh water. Even with the strongest *initial* density difference $\delta\rho$ (about 0·4) $\Delta\rho$ was less than 0·2 in the measurement phase of the experiment and the uncertainty here was less than 10 %. At first sight more serious is the use of

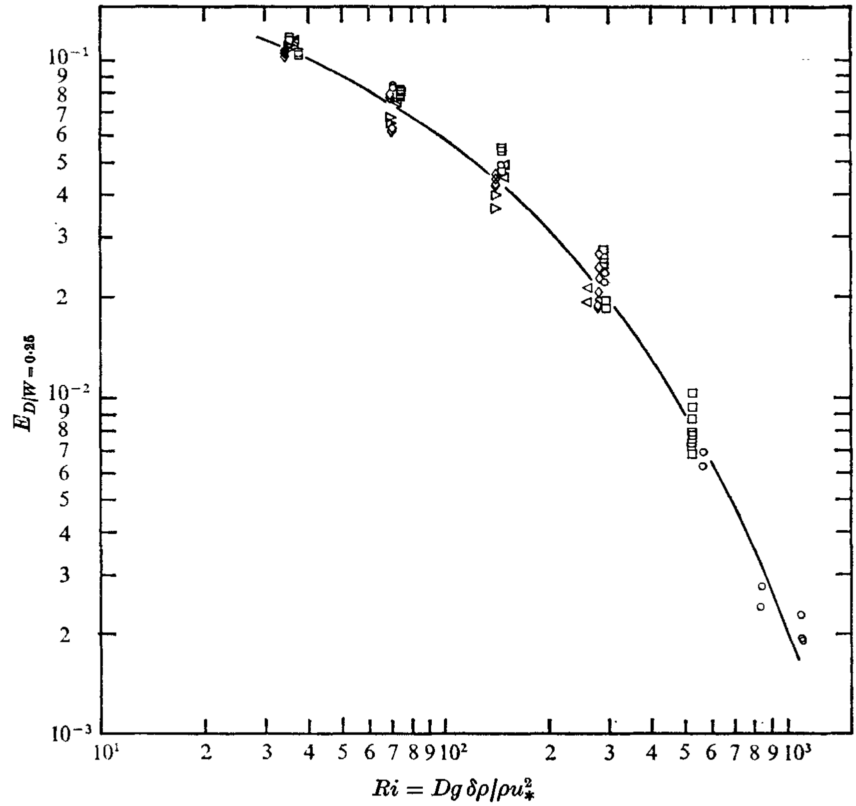

FIGURE 5. The variation of entrainment rate with Richardson number at a constant aspect ratio $D/W = 0{\cdot}25$. □, NaCl, $D_i = 5{\cdot}4$ cm, $u_* = 1{\cdot}41$ cm/s; ◇, NaCl, $D_i = 5{\cdot}4$ cm, $u_* = 1{\cdot}93$ cm/s; ◁, NaCl, $D_i = 2{\cdot}7$ cm, $u_* = 1{\cdot}41$ cm/s; ▷, NaCl, $D_i = 2{\cdot}7$ cm, $u_* = 1{\cdot}93$ cm/s; ○, $CaCl_2$, $D_i = 5{\cdot}4$ cm, $u_* = 1{\cdot}41$ cm/s.

the density ρ of fresh water in the definition of the Richardson number when $\delta\rho$ is as much as 0·4. If, however, this Richardson number is interpreted as the ratio of energy densities, the kinetic energy density in the upper layer which is responsible for entrainment is proportional to $(\rho+\Delta\rho)\,u_*^2$ rather than ρu_*^2; the difference leads to an uncertainty in the Richardson number of order 20 % and the variation in $\Delta\rho$ throughout an experiment also contributed to the variation in u_e/u_*.

In summary, the entrainment rate in the experiment is primarily a function of two variables,

$$E = u_e/u_* = f(Ri, D/W), \tag{3}$$

and we are interested in establishing the functional form of this relation particularly when $D/W \to 0$.

Figures 3 and 4 show a few of the measurements of layer depth D as a function of time for both NaCl and $CaCl_2$. At the lower Richardson numbers, when the interface was extensively convoluted and the experiment proceeded quickly, the position of the interface was recorded on movie film, the depth D being estimated by averaging over five or ten frames. This procedure reduced, but did not

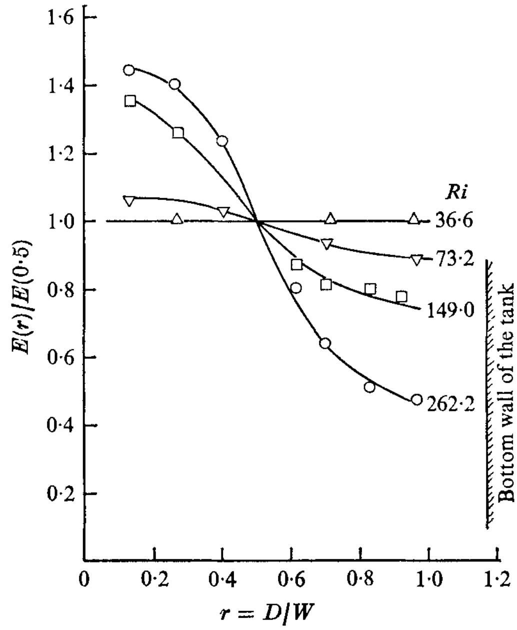

FIGURE 6. The entrainment rate at various Richardson numbers as a function of D/W, the stratification being produced by NaCl. The rates have been normalized by the values at $D/W = 0{\cdot}5$. $u_*^2 = 1{\cdot}41$ cm/s, $D_i = 2{\cdot}7$ cm.

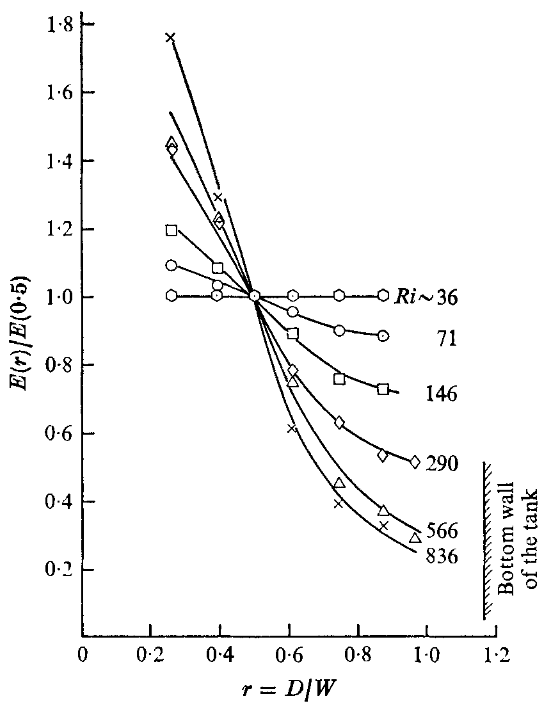

FIGURE 7. The entrainment rate at various Richardson numbers as a function of D/W, with stratification produced by $CaCl_2$. Again, the rate has been normalized by the values at $D/W = 0{\cdot}5$. $u_*^2 = 1{\cdot}41$ cm/s.

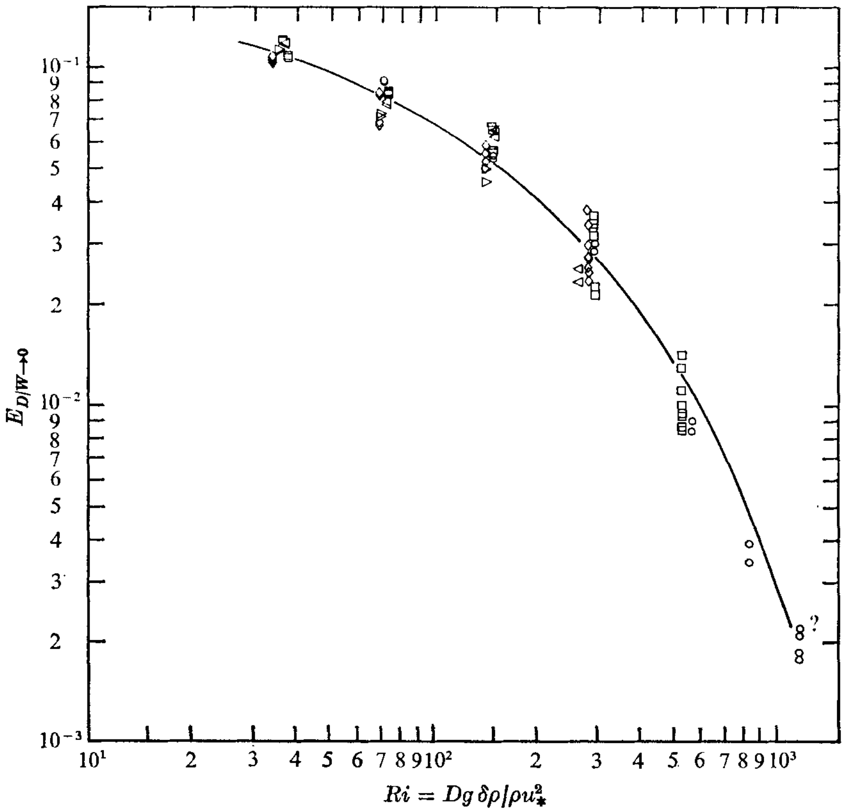

FIGURE 8. The entrainment rate as a function of Ri extrapolated to $D/W = 0$ from the data of which figures 6 and 7 are samples. The symbols are as specified in figure 5.

eliminate the experimental scatter. The flow is, of course, turbulent, so that random fluctuations must be expected. For Richardson numbers above about 300, the experiment was slower and the convolutions much less, so that the depth could be read and recorded visually to within $\pm 0{\cdot}2$ cm. Immediately after the start, the turbulence in the upper layer develops and within about 10 s entrainment begins. The subsequent increase of D with time is very nearly linear at least initially, when D/W is relatively small. When $Ri < 100$, there is little curvature in $D(t)$ as D/W increases and this suggests that the values measured are close to the limit $D/W \to 0$. At values of Ri approaching 1000, however, the slope decreases substantially with time, suggesting a more serious influence of side-wall drag and the need for larger correction.

Figure 5 shows the entrainment rate $E = u_e/u_*$ at a fixed aspect ratio D/W of 0·25. To obtain these points, the slopes of straight lines fitted to the data in this range were measured. Though there is a certain subjectivity in this, the errors in an individual determination did not appear to exceed 10 % or so and to give some confidence in the results, each experiment was repeated a number of times. In the calcium chloride measurements, the initial depth was $0{\cdot}25W$, so that for these points the initial slope was measured disregarding the first few points

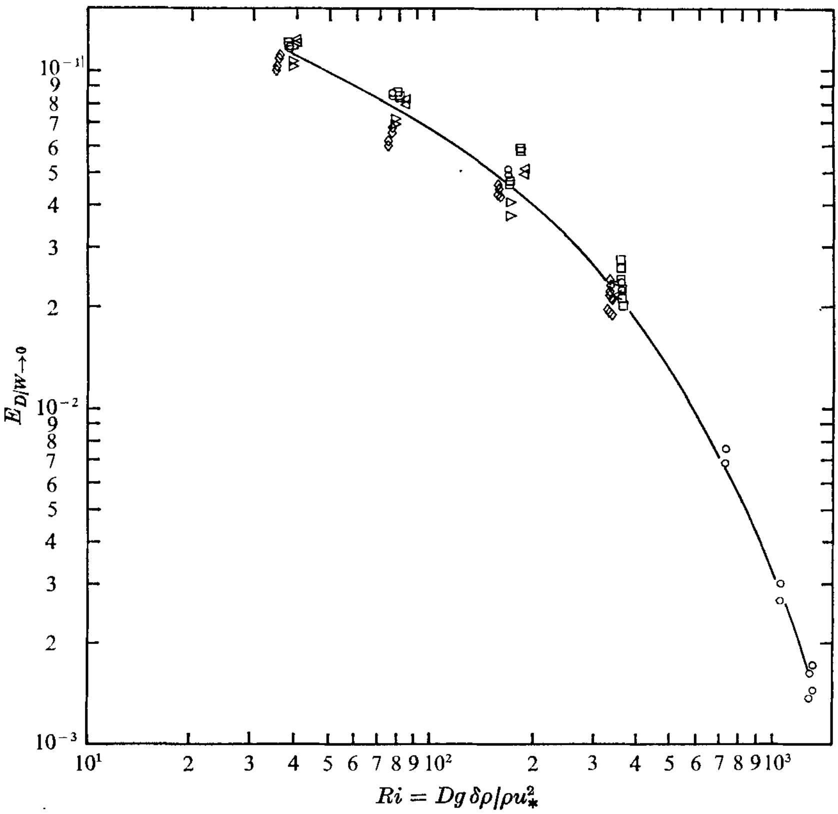

FIGURE 9. The entrainment rate as a function of Ri for $D/W = 0$, using the correction for side-wall drag given by Kantha (1975). The symbols are as specified in figure 5.

immediately following the start of an experiment. The range of E measured covered two decades, so that even uncertainties of $\pm 20\,\%$ will not, it is hoped, obscure the dependence on Ri.

Two methods were devised to estimate the limiting value of E as $D/W \to 0$. The first involved simple extrapolation. From data sets at different values of Ri, the slopes were estimated at various values of $r = D/W$ and the entrainment rates are shown as functions of r relative to that at $r = 0{\cdot}5$ in figure 6 for NaCl and figure 7 for $CaCl_2$. For the lower Richardson numbers there is very little dependence on r since $\dot{D}(t)$ is nearly constant throughout. For others, the straight line having the least-square error with respect to data points below $r = 0{\cdot}5$ was drawn through the point $r = 0{\cdot}5$ with ordinate unity and extrapolated to zero. This procedure in effect fits a parabola to the lower data points of sequences such as those shown in figures 3 and 4 and appears to be reasonably satisfactory except for the very highest Richardson number. It is evident from figure 4 that when $Ri = 1091$, the slope of the curve $D(t)$ decreases quite rapidly after a depth D of about 8 cm is attained, corresponding to $D/W \sim 0{\cdot}35$, and in this case it is difficult to do much better than simply take the initial slope. The results, corrected in this way, are shown in figure 8.

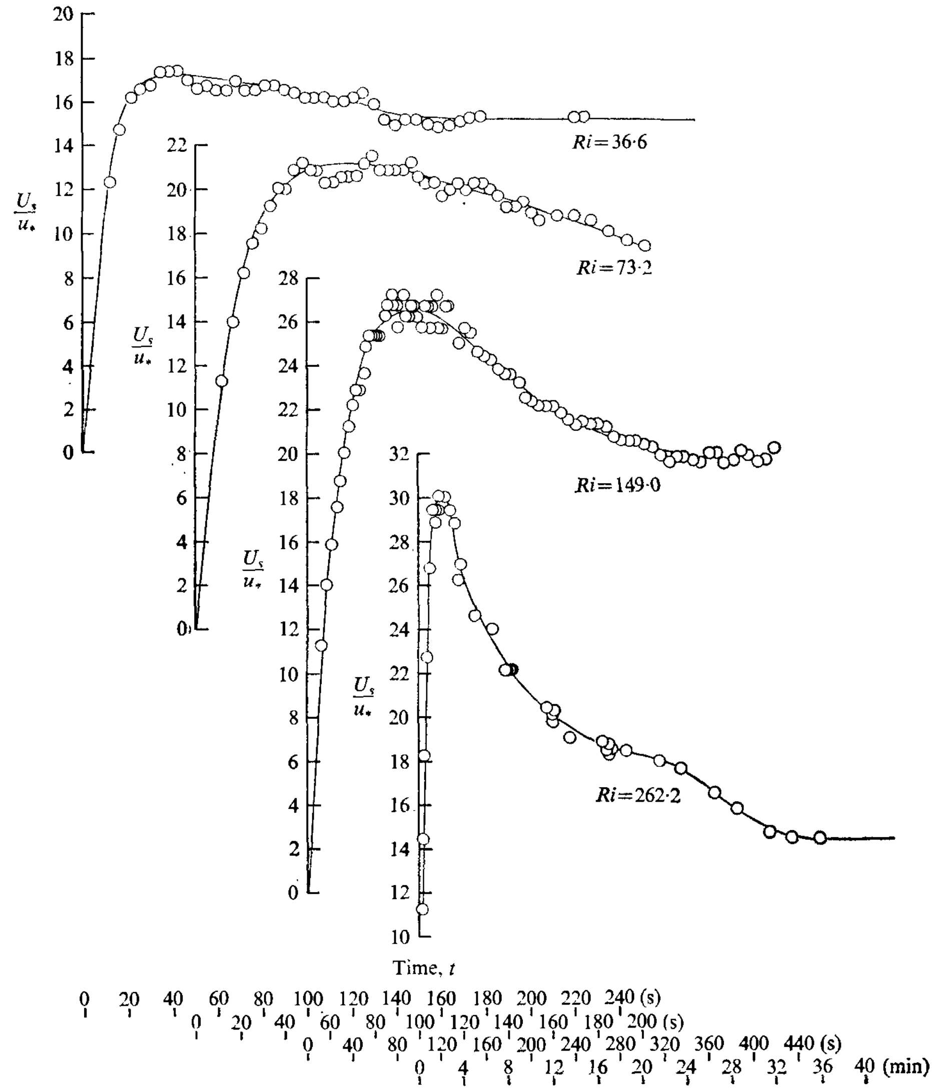

FIGURE 10. The screen velocity U_s/u_* as a function of time for various values of Ri in the experiments illustrated in figure 3. $u_* = 1{\cdot}41$ cm/s, $D_t = 2{\cdot}7$ cm.

An alternative method for estimating the entrainment rate as $r \to 0$ involves the assumption that the stress, represented by u_*^2, needed to parametrize the results is in fact the Reynolds stress in the turbulence just above the entrainment interface, which is less than that applied at the surface because of the drag of the side walls. A rough calculation using a friction coefficient of $0{\cdot}04R^{-\frac{1}{4}}$ and described in detail by Kantha (1975) provided a correction for u_* that increased with D/W and did not exceed 20 %. When this velocity is used to scale the measurements, the results are as shown in figure 9. The differences between this figure and figure 8 are probably not significant.

Two interesting conclusions emerge from these measurements. The first is that the entrainment rate evidently has no simple power-law dependence on Ri over the whole range studied. The slope of the measurements is like that of Ri^{-1} over the range $90 < Ri < 400$ but no reasonable (or even extreme) modification of the data analysis or subsequent correction can extend this range very much. Although at variance with earlier results on this apparatus and others from stirring experiments, this finding is hardly surprising in view of the different physical processes, described earlier, that are involved as the Richardson number increases. We are unable at present to devise any properly argued theory to account for the form of the relation $E = E(Ri)$.

The second point to note is that the entrainment rates measured here are approximately twice as large as those found by Kato & Phillips (1969) over the range $30 < Ri < 200$. In that experiment, the fluid below the mixed layer was linearly stratified, so that energy could be lost from the turbulent layer by radiation as internal waves in the fluid below. The reduced entrainment rate in these circumstances is consistent with a similar decrease found by Linden (1975).

In each experiment, the period of revolution T of the screen was also measured as a function of time, either by timing the wheel with a stop-watch or by determining from the movies the speed of marks made on the circumference of the wheel. The speed U_s of the screen, normalized by u_*, is shown in figure 10 as a function of time for the experiments shown in figure 3. The irregularities that can be seen in some experiments result mainly from the difficulty of keeping the angular position of the turntable constant. During the first few seconds of each experiment, the speed of the screen increases rapidly but by the time the entrainment is proceeding, the rate of variation is considerably less. At higher Richardson numbers, the speed of the screen required to maintain a given stress decreases considerably during the experiment as the depth of the mixed layer increases. We prefer not to speculate at this stage about the reasons for this effect.

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